On the Cycle Space of a 3–Connected Graph

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Abstract

We give a simple proof of Tutte's theorem stating that the cycle space of a 3–connected graph is generated by the set of non-separating circuits of the graph.

Keywords: graph, cycle, circuit, cycle space, non-separating circuit, strong isomorphism.

1 Introduction

We consider undirected graphs with no loops and no parallel edges. All notions on graphs that are not defined here can be found in [1, 8].

Let $G = (V, E, \psi)$ be a graph, where $V = V(G)$ is the set of vertices, $E = E(G)$ is the set of edges, and $\psi : E \to \binom{V}{2}$ is the edge-vertex incident function.

If C is a cycle of G then $E(C)$ is called a *circuit* of G. If $X, Y \subseteq E$, then let $X + Y$ denote the symmetric difference of X and Y, i.e. $X + Y = (X \cup Y) \setminus (X \cap Y)$. Then 2^E forms a vector space over $GF(2)$. Let $\mathcal{C}(G)$ denote the set of circuits of G, and so $\mathcal{C}(G) \subseteq 2^E$. Let $\mathcal{CS}(G)$ denote the subspace of 2^E generated by $\mathcal{C}(G)$. This subspace is called the *cycle space of* G. Obviously $X \in \mathcal{CS}(G)$ if and only if every vertex v in the subgraph of G induced by X has even degree. In particular, $\emptyset \in \mathcal{CS}(G)$. If $Z \subseteq E$, then let G/Z $(G \setminus Z)$ denote the graph obtained from G by contracting (respectively, deleting) the edges in Z. If A and B are subgraphs of G, we write, for simplicity, G/A instead of $G/E(A)$, $A + B$ instead of $E(A) + E(B)$, and $A \in \mathcal{F}$ instead of $E(A) \in \mathcal{F}$ for $\mathcal{F} \subseteq 2^E$.

A cycle C (the corresponding circuit $E(C)$) in a connected graph G is called *separating* if G/C has more blocks than G, and *non-separating*, otherwise. Let $\mathcal{NC}(G)$ denote the set of non-separating circuits of G, and so $\mathcal{NC}(G) \subseteq \mathcal{C}(G)$.

Given two graphs G and F with $E(G) = E(F)$, we say that G is *strongly isomorphic to* F if there is an isomorphism $v : V(G) \to V(F)$ from G to F that induces the identity map $\epsilon : E \to E$.

One of the classical Whitney theorems states:

1.1 [9] *Let G and F be two graphs such that $E(G) = E(F)$ and $\mathcal{C}(G) = \mathcal{C}(F)$. If G is 3–connected and F has no isolated vertices, then G is strongly isomorphic to F.*

A very simple proof of **1.1** is given in [2, 3].

In [2] we proved the following strengthening of **1.1**.

1.2 *Let G and F be two graphs such that $E(G) = E(F)$ and $\mathcal{NC}(G) = \mathcal{NC}(F)$. If G is 3–connected and F has no isolated vertices, then G is strongly isomorphic to F.*

In [5] we gave some other strengthenings of the Whitney theorem **1.1**.

The following theorem, due to W. Tutte [7] and, independently, A. Kelmans [2, 3], is an important result in the study of the graph cycle spaces.

1.3 *The set of non-separating circuits of a 3–connected graph generates the cycle space of the graph.*

The above Theorem is an obvious Corollary of **1.2**. On the other hand, **1.2** follows from **1.1** and **1.3**.

In [2] we proved the following theorem.

1.4 *Suppose that G is a 3–connected graph, $X \subseteq E(G)$ and $G \setminus X$ is a connected graph. Then there exist two distinct non-separating circuits A, B in G such that $|A \cap X| = 1$ and $|B \cap X| = 1$.*

We also gave the following simple

Proof of **1.2**, and therefore also **1.3**, using **1.4** [2]. Let G be a 3-connected graph. It is sufficient to show that the set $\mathcal{K}(G)$ of cocircuits (i.e. minimal edge cuts) of G is uniquely defined by the set $\mathcal{NC}(G)$ of non-separating circuits of G. Let $\mathcal{K}'(G)$ be the set of edge subsets X of G such that $X \neq \emptyset$ and $|X \cap C| \neq 1$ for every $C \in \mathcal{NC}(G)$. Obviously $\mathcal{K}(G) \subseteq \mathcal{K}'(G)$. Let $\mathcal{K}''(G)$ be the set of members of $\mathcal{K}'(G)$ minimal by inclusion. By **1.4** , if $X \in \mathcal{K}'(G)$, then there is $Y \in \mathcal{K}(G)$ such that $Y \subseteq X$. Since $Y \in \mathcal{K}(G)$, every proper subset of Y is not in $\mathcal{K}(G)$. Therefore
$\mathcal{K}(G) \subseteq \mathcal{K}'(G) \Rightarrow \mathcal{K}''(G) = \mathcal{K}(G)$. □

There are several other proofs of **1.3** (see, for example, [1, 8]).

In this paper we give a new fairly simple proof of **1.3**.

The results of this paper were presented at the Moscow Discrete Mathematics Seminar in 1977 (see also [6]).

2 Proof of 1.3

We call a graph *topologically 3–connected*, or simply *top 3–connected*, if it is a subdivision of a 3–connected graph. A subdivision of a graph G is called *top* G.

A *thread* in G is a path T in G such that the degree of every inner vertex of T is equal to two and the degree of every end-vertex of T is not equal to two in G. Obviously if C is a cycle of G and $E(C) \cap E(T) \neq \emptyset$, then $T \subseteq C$. If T is a thread in G, we write $G - (T)$ instead of $G - (T - End(T))$.

A path P with end-vertices x and y is called a *path-chord of* a cycle C in G if $V(C) \cap V(P) = \{x, y\}$, and $E(C) \cap E(P) = \emptyset$.

We need the following known facts.

2.1 [3] *Let G be a top 3–connected graph and G not top K_4. Then G has a thread T such that $G-(T)$ is also a top 3–connected graph.*

2.2 [3] *Let G be a top 3–connected graph, C a cycle of G, and T a thread of G which is a path-chord of C, and let R, S be the cycles of $C \cup T$ distinct from C. If C is a non-separating cycle of $G-(T)$, then R and S are non-separating cycles of G.*

Proof. Let $Q = S-(T)$. Then G/R has a block, say H, containing $E(Q)$. Suppose, on the contrary, that $R \notin \mathcal{NC}(G)$, i.e. G/R has a block B distinct from H. Then B is also a block of G/C. Suppose that $E(H) \neq E(Q)$. Let P be a block of G/C that meets $E(H) \setminus E(Q)$. Then $E(P) \neq E(B)$ and $E(P) \neq E(T)$, and therefore $C \notin \mathcal{NC}(G-(T))$, a contradiction. Thus $E(H) = E(Q)$. Then Q is a thread of G and Q is parallel to T. Therefore G is not top 3–connected, a contradiction. □

2.3 [2, 3] *Let G be a 3–connected graph. Then for every edge e of G there are two non-separating cycles P and Q of G such that $E(P) \cap E(Q) = e$ and $V(P) \cap V(Q) = \psi(e)$.*

Proof (a sketch). Since G is top 3–connected, there are two cycles R and S such that $R \cap S = T$. Let $\mathcal{C}_R$ be the set of cycles C in G such that $C \cap R = T$, and so $S \in \mathcal{C}_R$. If $C \in \mathcal{C}_R$, then let $\alpha(C)$ be the number of edges of the block of G/C containing $E(R-(T))$. Let P be a cycle in $\mathcal{C}_R$ such that $\alpha(P) = \max\{\alpha(C) : C \in \mathcal{C}_R\}$. It is easy to show that P is a non-separating cycle of G.

Applying the above arguments to $R := P$ and $S := R$, we find another non-separating cycle Q of G such that $P \cap Q = T$. □

Now we are ready to prove the following equivalent of **1.3**.

2.4 *Let G be a top 3–connected graph. Then $\mathcal{CS}(G)$ is generated by $\mathcal{NC}(G)$.*

Proof (uses **2.1**, **2.2**, and **2.3**). We prove our claim by induction on the number $t(G)$ of threads of G. If G is top K_4, then our claim is obviously true. So let $t(G) \geq 7$. By **2.1**, G has a thread T such that $G' = G-(T)$ is top 3–connected. By the induction hypothesis, $\mathcal{CS}(G')$ is generated by $\mathcal{NC}(G')$. Obviously if $Q \in \mathcal{NC}(G')$ and T is not a path-chord of Q, then $Q \in \mathcal{NC}(G)$. By **2.2**, if $C \in \mathcal{NC}(G')$, T is a path-chord of C, and R, S are the cycles of $C \cup T$ distinct from C, then $R, S \in \mathcal{NC}(G)$. In this case $C = R+S$. Therefore every cycle in G' is generated by $\mathcal{NC}(G)$. Now let A be a cycle in G but not in G'. Then $T \subseteq A$. By **2.3**, there are $P, Q \in \mathcal{NC}(G)$ such that $P \cap Q = T$. Since $T \subseteq A$ and $T \subseteq P$, clearly $A+P \in \mathcal{CS}(G')$, and so $A+P$ is generated by $\mathcal{NC}(G)$. Since $(A+P)+P = A$ and $P \in \mathcal{NC}(G)$, clearly A is also generated by $\mathcal{NC}(G)$. □

More information on this topic can be found in the expository paper [4].

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